AIAA Journal
Vol. 47, No. 10, October 2009

Technical Note

Reinvestigation of Intuitive Approach for Thermal Postbuckling of Circular Plates

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DOI: 10.2514/1.43034

Nomenclature

a	=	radius of the circular plate
b	=	central (maximum) lateral deflection of the circular plate
C	=	ratio of λ_T/λ_{cr}
D	=	plate flexural rigidity, $Et^3/12(1-\nu^2)$
E	=	Young's modulus
N_{cr}	=	linear buckling load
T_r	=	radial tensile load per unit length (from $r=0$ to $r=a$) developed due to large lateral displacements
t	=	thickness of the circular plate
u	=	radial displacement
w	=	lateral displacement
γ	=	ratio of the postbuckling to linear buckling load parameters
ΔT	=	temperature rise from the stress-free temperature
ε_r, ε_θ	=	in-plane strains
λ_{cr}	=	critical buckling load parameter
λ_{PB}	=	radial tensile load parameter
λ_T	=	radial tensile load parameter
ν	=	Poisson's ratio

Introduction

STUDY of thermal postbuckling behavior of structural members such as columns and plates subjected to thermal loads is necessary in the design of aerospace structures, as one can use the postbuckling load-carrying capacity for obtaining efficient designs. It has also been shown through rigorous continuum and finite element (FE) formulations in the works of Thompson and Hunt [1], Dym [2], Rao and Raju [3,4], and Raju and Rao [5] that the thermal postbuckling load-carrying capacity of the aforementioned structural members is an order of magnitude higher than the load-carrying capacity subjected to mechanical loads. Recently, this complex thermal postbuckling phenomenon has been investigated through intuitive formulations that give simple and reliable closed-form solutions [6,7]. The intuitive formulations require the knowledge of the linear buckling load and the tension developed because of the large lateral deflections of these structural members. As shown in [6,7], it is a simple and straightforward procedure to obtain the tension for columns and square plates. However, when dealing with the circular plates, the evaluation of the radial tension is not that simple, but poses problems because of the explicit coupling in the expression for the radial and circumferential strains due to the radial displacement. Because of this coupling, some assumptions or approximations have to be made to evaluate the radial tension. In [8], the assumption of zero Gaussian curvature of the deflected plate is used to evaluate the radial tension, and this assumption gives satisfactory postbuckling results in terms of the ratios of postbuckling load parameter to the critical load parameter for clamped circular plates, but it gives lower values for the simply supported circular plates when compared with the solutions obtained by using the FE method [5]. As a result, several approximate solutions have been developed [9], and all of these approximations give more or less satisfactory results for the clamped circular plate, with wide variation of results for the simply supported circular plate. In [9], in one of the approximations, the tension obtained by using Berger's approximation [10] gives constant tension [11,12] and results in very high values for the simply supported circular plate. The assumption that the radial displacement varies linearly with the radial coordinate gives more or less satisfactory results for both boundary conditions. This anomalous behavior of the simple formulations developed for the circular plate motivated the authors to further investigate its thermal postbuckling behavior. In the proposed investigation, Berger's approximation [10] is used to evaluate the functional form of radial displacement once the lateral displacement distribution satisfying all the boundary conditions is known. The radial tension derived by using Berger's approximation [10] is treated as constant, as in [11]. In the following section, the proposed formulation to obtain the radial tension is briefly discussed, highlighting the basic features.

Evaluation of Radial Tension

Consider a circular plate of radius a with edges immovable in the radial direction (Fig. 1). If the plate is heated to a temperature ΔT from the stress-free state, an equivalent uniform compressive edge load N_r is developed. If the plate undergoes large deflections in the postbuckling state, a radial tensile load is developed that, in general, varies along the radius. The radial tension can be evaluated using the nonlinear strain-displacement relations for the axisymmetric case, neglecting the small initial geometric imperfections, given by

$$\varepsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 \tag{1}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{2}$$

Note that ε_r and ε_θ are coupled by the radial displacement u, and this makes the evaluation of tension parameter difficult for the circular plates, which is not the situation in the case of square plates [7]. From Eqs. (1) and (2), the radial tension per unit length is obtained as

$$T_r = \frac{Et}{(1-\nu^2)}[\varepsilon_r + \nu\varepsilon_\theta] \tag{3}$$

With the approximation proposed by Berger [10], the second strain invariant $\varepsilon_r \cdot \varepsilon_\theta$ is neglected, which implies that $\varepsilon_\theta \ll \varepsilon_r$. T_r is rewritten after substituting the expression for ε_r, as

Received 2 January 2009; revision received 26 June 2009; accepted for publication 6 July 2009.

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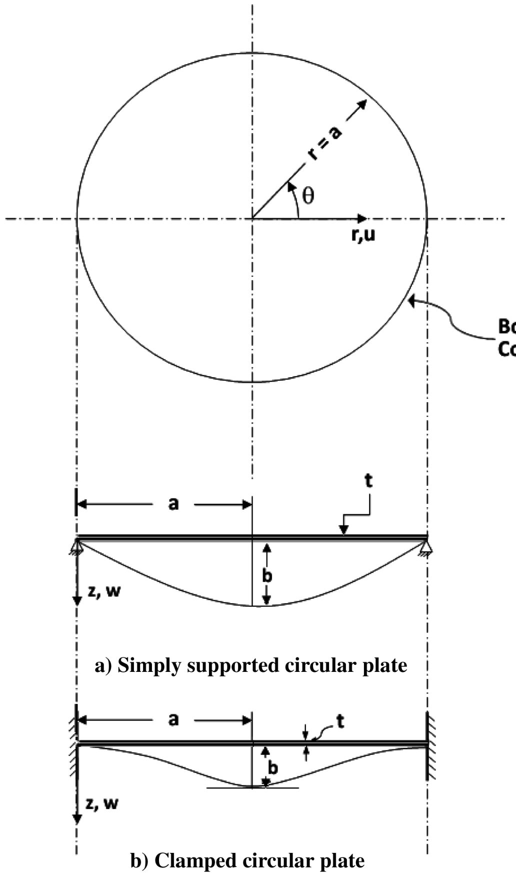

Fig. 1 Circular plate showing coordinate system and lateral deflection pattern.

$$T_r = \frac{Et}{(1-\nu^2)}\left[\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2\right] \tag{4}$$

Following the work of Wah [11] given by Leissa [12], the tensile load T_r is treated constant, and hence

$$\frac{\mathrm{d}T_r}{\mathrm{d}r} = 0 \tag{5}$$

or

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} = -\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)\left(\frac{\mathrm{d}^2w}{\mathrm{d}r^2}\right) \tag{6}$$

For the chosen admissible function for w, the functional form of u can be evaluated by integrating Eq. (6) twice. The constants of integration are obtained using the boundary conditions on radial displacement.

Once the functional form of u is known, a better approximation for the radial tension T_r, which is still treated as constant, along the radius is obtained as

$$T_r = \frac{Et}{(1-\nu^2)}\left[\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 + \nu\left(\frac{u}{r}\right)_{\mathrm{av}}\right] \tag{7}$$

Note that in Eq. (7), to make T_r constant along the radius, the integrated average of $\varepsilon_\theta = (u/r)_{\mathrm{av}}$ is given by

$$\left(\frac{u}{r}\right)_{\mathrm{av}} = \varepsilon_{\theta\mathrm{av}} = \frac{\int_0^a \frac{u}{r}\,\mathrm{d}r}{a} \tag{8}$$

Numerical Results and Discussion

From Eq. (7), the uniform radial edge tension T_r developed in the circular plate with edges immovable in the radial direction can be obtained by assuming suitable admissible functions for the lateral displacement w. The value of Poisson's ratio ν is taken as 0.3 for obtaining the numerical results. Both the simply supported and clamped boundary conditions of the circular plate are considered. The admissible function for the lateral displacement is taken in the form summation of n functions, taking into account each term that satisfies both the geometric and natural boundary conditions. Each term considered for w is of the form given by Yamaki [13] and the series taken for w is

$$w = \sum_{i=0}^{n} b\left[\left(\frac{r}{a}\right)^{2i} + \alpha_{2i+1}\left(\frac{r}{a}\right)^{2i+2} + \alpha_{2i+2}\left(\frac{r}{a}\right)^{2i+4}\right] \tag{9}$$

The values of α_{2i+1} and α_{2i+2} can be obtained from the boundary conditions of the plate. The radial displacement u for each term of the w series is obtained as explained in the previous section.

The proposed methodology to obtain the value of the tension induced in the plate due to large axisymmetric lateral displacements is given in previous section. The simple formulations to obtain the thermal postbuckling [6,7] behavior is given in terms of λ_T and λ_{cr} as

$$\gamma = \frac{\lambda_{\mathrm{PB}}}{\lambda_{\mathrm{cr}}} = 1 + \frac{\lambda_T}{\lambda_{\mathrm{cr}}}\left(\frac{b_0}{t}\right)^2 \tag{10}$$

where the nondimensional parameters λ_T and λ_{cr} are defined as

$$\lambda_T = \frac{T_r a^2}{D} \tag{11}$$

$$\lambda_{\mathrm{cr}} = \frac{N_{\mathrm{cr}} a^2}{D} \tag{12}$$

Table 1 Variation of γ for uniform thin circular plates with b/t

	Simply supported				Clamped			
	Present study				Present study			
b/t	One term	Two term	Three term	FEM [5]	One term	Two term	Three term	FEM [5]
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0144	1.0175	1.0190	1.0182	1.0046	1.0050	1.0052	1.0052
0.2	1.0574	1.0703	1. 0706	1.0730	1.0184	1.0201	1.0208	1.0210
0.3	1.1293	1.1581	1.1711	1.1645	1.0415	1.0452	1.0468	1.0472
0.4	1.2299	1.2812	1.3043	1.2931	1.0738	1.0804	1.0832	1.0840
0.5	1.3593	1.4394	1.4755	1.4593	1.1154	1.1256	1.1300	1.1314
0.6	1.5173	1.6327	1.6847	1.6637	1.1662	1.1809	1.1873	1.1893
0.7	1.7042	1.8612	1.9320	1.9072	1.2262	1.2462	1.2549	1.2580
0.8	1.9198	2.1249	2.2174	2.1907	1.2955	1.3216	1.3330	1.3373
0.9	2.1641	2.4237	2.5407	2.5151	1.3740	1.4070	1.4215	1.4275
1.0	2.4372	2.7576	2.9022	2.8818	1.4617	1.5025	1.5203	1.5286
λ_{cr}	5.2000	4.1977	4.1978	4.1978	16.0000	14.7017	14.6820	14.6896
C	1.4372	1.7576	1.9022	1.8818	0.4617	0.5025	0.5203	0.5286

The convergence on the number of terms used in the lateral deflection distribution w is studied, which is achieved by choosing three terms in the w distribution, based on the small percentage error for the values of buckling load parameter and of γ at the maximum value of $b_0/t = 1.0$. The values of γ and $C = \lambda_T/\lambda_{cr}$ for the simply supported and clamped boundary conditions of the circular plate with the condition of radially immovable edges for different values of b_0/t are given in Table 1. The results obtained by the present formulation and those given by the FE method [5] match well, with a percentage difference of -1.30 to 0.22%, as shown in the table. Please note that in the present study, the emphasis is given on the evaluation of λ_T, whereas λ_{cr} values are obtained from the admissible functions chosen for w or, alternatively, one can obtain the values of λ_{cr} from the available literature [14,15].

Conclusions

The proposed simple formulation to obtain the thermal postbuckling behavior of circular plates resolves the anomaly existing in the similar formulations developed earlier. The results in terms of the ratios of the postbuckling to the buckling load match very well with those obtained by the FE method, irrespective of the edge boundary conditions on the lateral displacements. With the very encouraging results obtained from the present investigation, the authors propose to work on prediction of the thermal postbuckling behavior of circular plates with complicating effects such as the effect of elastic foundation, moderately thick circular plates, circular plates with elastic rotational edge restraints, etc.

Acknowledgments

The authors are grateful to the management of Sreenidhi Institute of Science & Technology (SNIST) for their constant encouragement during the course of this work. The authors are also thankful to the University Grants Commission of India and the Indian National Academy of Engineering for awarding the Junior Research Fellowship and Distinguished Professorship, respectively, tenable at SNIST.

References

[1] Thompson, J. M. T., and Hunt, G. W., *A General Theory of Elastic Stability*, Wiley, London, 1973.

[2] Dym, C. L., *Stability Theory and its Applications to Structural Mechanics*, Noordhoff International, Leyden, The Netherlands, 1974.

[3] Rao, G. V., and Raju, K. K., "A Reinvestigation of Postbuckling Behaviour of Elastic Circular Plates Using a Simple Finite Element Formulation," *Computers and Structures*, Vol. 17, No. 2, 1983, pp. 233–235.
doi:10.1016/0045-7949(83)90011-1

[4] Rao, G. V., and Raju, K. K., "Thermal Post-Buckling of Columns," *AIAA Journal*, Vol. 22, No. 6, 1984, pp. 850–851.

[5] Raju, K. K., and Rao, G. V., "Thermal Post-Buckling of Circular Plates," *Computers and Structures*, Vol. 18, No. 6, 1984, pp. 1179–1182.
doi:10.1016/0045-7949(84)90162-7

[6] Rao, G. V., and Raju, K. K., "Thermal Post-Buckling of Uniform Columns: A Simple Intuitive Method," *AIAA Journal*, Vol. 40, No. 10, 2002, pp. 2138–2140.
doi:10.2514/2.1553

[7] Rao, G. V., and Raju, K. K., "Applicability of a Simple Method for Thermal Post Buckling of Square Plates," *AIAA Journal*, Vol. 42, No. 8, 2004, pp. 1724–1726.

[8] Rao, G. V., and Varma, R. R., "Simple Formulation to Predict Thermal Post-Buckling Load of Circular Plates," *AIAA Journal*, Vol. 45, No. 7, 2007, pp. 1784–1786.
doi:10.2514/1.29021

[9] Varma, R. R., and Rao, G. V., "Thermal Postbuckling Behavior of Circular Plates—Study on Various Approximations," *Journal of Structural Engineering*, Vol. 26, No. 2, June 2009, pp. 148–151.

[10] Berger, H. M., "A New Approach to the Analyses of Large Deflection of Plates," *Journal of Applied Mechanics*, Vol. 22, No. 4, Dec. 1955, p. 465.

[11] Wah, T., " Vibration of Circular Plates at Large Amplitudes," *Journal of the Engineering Mechanics Division, American Society of Civil Engineers*, Vol. 89, No. EM5, Oct. 1963, pp. 1–15.

[12] Leissa, A. W., "Vibration of Plates," NASA SP-160, Washington, D.C., 1965.

[13] Yamaki, N., "Influence of Large Amplitudes on Flexural Vibrations of Elastic Plates," *ZAMM*, Vol. 41, No. 12, 1961, pp. 501–510.
doi:10.1002/zamm.19610411204

[14] Timoshenko, S. P., and Gere, J. M., *Theory of Elastic Stability*, 2nd ed., McGraw–Hill, Tokyo, 1961.

[15] *Handbook of Structural Stability*, edited by Column Research Committee of Japan, Corona, Tokyo, 1971.

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